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Closure result for Γ-limits of functionals with linear growth

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Abstract
We consider integral functionals $\mathcal{F}_\varepsilon^{(j)}$, doubly indexed by $\varepsilon > 0$ and $j \in \mathbb{N} \cup \{\infty\}$, satisfying a standard linear growth condition. We investigate the question of Γ-closure, i.e., when the Γ-convergence of all families $\{\mathcal{F}_\varepsilon^{(j)}\}_\varepsilon$ with finite j implies Γ-convergence of $\{\mathcal{F}_\varepsilon^{(\infty)}\}_\varepsilon$. This has already been explored for p-growth with $p > 1$. We show by an explicit counterexample that due to the differences between the spaces $W^{1,1}$ and $W^{1,p}$ with $p > 1$, an analog cannot hold. Moreover, we find a sufficient condition for a positive answer.

Keywords Γ-convergence · Γ-closure results · Functionals with standard linear growth

Mathematics Subject Classification 49J45 (74Q05)

1 Introduction

A possibility to formulate a mathematical problem is by means of calculus of variations. The solution has to minimize a given functional. Often, functionals are of the form

$$\mathcal{F}(u) := \begin{cases} \int_\Omega f(x, \nabla u(x))\, \mathrm{d}x, & u \in W^{1,p}(\Omega; \mathbb{R}^m), \\ \infty, & u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(\Omega; \mathbb{R}^m). \end{cases} \tag{1}$$

with Ω being an open bounded subset of $\mathbb{R}^n$ and $1 \le p \le \infty$. The choice of the ambient space $L^p(\Omega)$ instead of the domain $W^{1,p}(\Omega)$ is motivated by the direct method and good properties of the weak topology of $W^{1,p}(\Omega)$.

Depending on the model, the density f of a functional may be quite complicated. It is desirable, if possible, to find a good efficient model, i.e., a functional that is simpler but, however, whose minimizers are good approximations of the solutions of the original functional.

In order for an efficient model to have the desired properties, the right notion of convergence is Γ-convergence. We state some fundamental definitions and results regarding this concept in Appendix. There is an extensive literature, let us mention only the works [9, 12].

A typical example of such efficient model is homogenization. Having a heterogeneous material with regular structure (e.g., periodic) on a small scale, it is to be expected that this

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material behaves on the macroscopic level as some theoretical homogeneous material. A homogenization procedure for a periodic stored energy function of the material dates back to the works [6, 17].

Suppose two models are in some sense close to each other. If one of them can be homogenized, then one would expect the same for the other one. This was addressed and covered, e.g., in [7, Section 3] and [8, Section 4] as ‘homogenization closure’. An example can be found in [18] where the model for finite elasticity is approximated by a linearized one.

In [16], instead of specializing to homogenization, an arbitrary Γ-converging family was taken. The setting can be summarized with the following diagram:

$$\begin{array}{ccc} \mathcal{F}_\varepsilon^{(j)} & \xrightarrow{\;?\;} & \mathcal{F}_\varepsilon^{(\infty)} \\ {\scriptstyle\Gamma}\big\downarrow & & \\ \mathcal{F}_0^{(j)} & & \end{array}$$

For each $j \in \mathbb{N}$ the family $\{\mathcal{F}_\varepsilon^{(j)}\}_{\varepsilon>0}$ is supposed to Γ-converge as $\varepsilon \to 0$. These families converge in some sense to a family $\{\mathcal{F}_\varepsilon^{(\infty)}\}_{\varepsilon>0}$ as $j \to \infty$. The authors investigate when this family Γ-converges as well and what is the relation between the Γ-limits. Let us below briefly present their assumptions and results.

Assumption Throughout the whole work, we assume:

- $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz open set.
- Each function $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a Carathéodory function.

Regarding the densities of functionals, the following property is assumed.

Definition 1.1 Function $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ fulfills a *standard p-growth condition* if there are $\alpha, \beta > 0$ such that

$$\alpha|X|^p - \beta \le f(x, X) \le \beta(|X|^p + 1)$$

for almost every $x \in \Omega$ and all $X \in \mathbb{R}^{m\times n}$. (For $p = 1$, we also say a standard linear growth condition.) For a family of functions, we say that they fulfill the standard p-growth condition *uniformly* if the double inequality above holds for all members of the family with the same α and β.

For the proximity of families, it turns out that the following generalization of equivalence introduced in [7] is the right one.

Definition 1.2 Let us have functions $f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ for all $j \in \mathbb{N} \cup \{\infty\}$ and $\varepsilon > 0$. We say that the a doubly indexed family $\{f_\varepsilon^{(j)}\}_{\varepsilon>0}^{j\in\mathbb{N}}$ is *equivalent* to a family $\{f_\varepsilon^{(\infty)}\}_{\varepsilon>0}$ if

$$\lim_{j\to\infty} \limsup_{\varepsilon\to 0} \int_\Omega \sup_{|X|\le R} |f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)| \,\mathrm{d}x = 0$$

for every $R \ge 0$.

We are not in the position to state [16, Theorem 2.2]:

Theorem 1.3 *Let* $1 < p < \infty$*, and let us have functions* $f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$*,* $j \in \mathbb{N} \cup \{\infty\}$*,* $\varepsilon > 0$*, that uniformly fulfill a standard* p*-growth condition. We define the corresponding functionals* $\mathcal{F}_\varepsilon^{(j)}$ *on* $L^p(\Omega; \mathbb{R}^m)$ *as in* (1) *and assume that*

(a) *For each* $j \in \mathbb{N}$ *the* Γ*-limit* $\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(j)} =: \mathcal{F}_0^{(j)}$ *exists and that*
(b) *The families* $\{f_\varepsilon^{(j)}\}_{\varepsilon>0}^{j\in\mathbb{N}}$ *and* $\{f_\varepsilon^{(\infty)}\}_{\varepsilon>0}$ *are equivalent on* Ω*.*

Then also $\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\infty)} =: \mathcal{F}_0^{(\infty)}$ *exists. It is the pointwise and the* Γ*-limit of* $\mathcal{F}_0^{(j)}$ *as* $j \to \infty$*:*

$$\mathcal{F}_0^{(\infty)} = \lim_{j\to\infty} \mathcal{F}_0^{(j)} = \Gamma(L^p)\text{-}\lim_{j\to\infty} \mathcal{F}_0^{(j)}.$$

Schematically,

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathcal{F}_\varepsilon^{(j)} \arrow[r, equal, "\approx"] \arrow[d, "\Gamma"'] & \mathcal{F}_\varepsilon^{(\infty)} \arrow[d, dashed, "\Gamma"] \\
\mathcal{F}_0^{(j)} \arrow[r, dashed, "\textit{pointwise}", "\Gamma"'] & \mathcal{F}_0^{(\infty)}
\end{tikzcd}$$

In [16] this result is called ‘Γ-closure on a single domain’ since also the complementing result for variable domains, i.e., for functionals that depend also on the domain, is proved. Moreover, more general growth conditions, the Gårding growth conditions, are explored, and the question about simultaneous limits is addressed. Great emphasis is then laid on applying this results to homogenization, also in the stochastic setting. We mention below only one special case, which will be important for a current work, namely the perturbation result [16, Theorem3.1].

Let us have a situation as in Theorem 1.3 but suppose, however, that the family $\{f_\varepsilon^{(j)}\}_{\varepsilon>0}^{j\in\mathbb{N}}$ is actually constant in j. In other words, let us have families $\{f_\varepsilon\}_{\varepsilon>0}$ and $\{g_\varepsilon\}_{\varepsilon>0}$ that are equivalent. This condition now reads

$$\limsup_{\varepsilon\to 0} \int_\Omega \sup_{|X|\le R} |f_\varepsilon(x, X) - g_\varepsilon(x, X)| \,\mathrm{d}x = 0$$

for every $R > 0$. If the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ with densities f_ε $\Gamma(L^p)$-converges to a functional $\mathcal{F}_0$ as $\varepsilon \to 0$, then so does the analogously defined family $\{\mathcal{G}_\varepsilon\}_{\varepsilon>0}$:

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}_0 = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon.$$

The results in [16] for $p > 1$ lead to a natural question if the same holds for $p = 1$. The answer is negative, as we will show with an explicit counterexample, the problem being that in $W^{1,1}$ boundedness of gradients does not imply their equiintegrability. As already known from the literature, in this case the behavior at ∞, described by the recession function, also must be taken into account. Therefore, for analogous commutability results an additional assumption is needed.

Our aim is not to find a counterpart to all results in [16]. We will only look at the basic deterministic setting from Theorem 1.3 since the proof already contains the most important ideas.

2 Counterexample

For $p = 1$, integral functionals with convex densities defined as in (1) are not lower semi-continuous as is the case for $p > 1$. More precisely, for a convex function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ with a standard linear growth, the lower semicontinuous envelope of the functional

$$\mathcal{F}(u) = \begin{cases} \int_\Omega f(\nabla u(x))\,\mathrm{d}x, & u \in W^{1,1}(\Omega;\mathbb{R}^m),\\ \infty, & u \in L^1(\Omega;\mathbb{R}^m)\setminus W^{1,1}(\Omega;\mathbb{R}^m), \end{cases}$$

is

$$\mathcal{F}(u) = \begin{cases} \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{d(D^s u)}{d|D^s u|}(x)\right)\mathrm{d}|D^s u|(x), & u \in BV(\Omega;\mathbb{R}^m),\\ \infty, & u \in L^1(\Omega;\mathbb{R}^m)\setminus BV(\Omega;\mathbb{R}^m). \end{cases} \tag{2}$$

This was first proved in [15, Theorem 5]. It actually holds in this form for quasiconvex f, see [1, Theorem 4.1] and [14, Theorem 2.16]. Here, $BV(\Omega)$ is the space of functions with bounded variation (see, e.g., [2]), and Du is the distributional derivative of u having decomposition with respect to the Lebesgue measure $Du = \nabla u \mathcal{L}^n + D^s u$. Moreover, f^∞ is the recession function f:

$$f^\infty(\xi) = \lim_{t\to\infty} \frac{f(t\xi)}{t}.$$

For the counterexample we employ [4, Example 6.4]. We consider the scalar case $m = 1$ with $\Omega = J := (-1, 1)$ and show that even the analog to the perturbation result described above ([16, Theorem 3.1]) does not hold.

Let us choose the convex function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(\xi) := \max\{|\xi|, 2|\xi| - 1\}.$$

Clearly, $f^\infty(\xi) = \lim_{t\to\infty} \frac{f(t\xi)}{t} = 2|\xi|$.

For the first family of functions, we choose the constant family given by f, i.e., for all $\varepsilon > 0$ let $f_\varepsilon(x, \_) := f$ for all $x \in J$. The corresponding constant family of functionals is thus

$$\mathcal{F}_\varepsilon(u) := \mathcal{F}(u) := \begin{cases} \int_\Omega f(u'(x))\,\mathrm{d}x, & u \in W^{1,1}(J),\\ \infty, & u \in L^1(J)\setminus W^{1,1}(J). \end{cases} \tag{3}$$

It $\Gamma(L^1)$-converges to the relaxed functional $\mathcal{F}_0$ with domain $BV(J)$. By (2), for $u \in BV(J)$ we have

$$\begin{aligned} \mathcal{F}_0(u) &= \int_{-1}^1 f(u'(x))\,\mathrm{d}x + \int_{-1}^1 f^\infty\left(\frac{\mathrm{d}(D^s u)}{\mathrm{d}|D^s u|}(x)\right)\mathrm{d}|D^s u|(x)\\ &= \int_{-1}^1 f(u'(x))\,\mathrm{d}x + 2|D^s u|(J). \end{aligned}$$

For the second family let

$$a_\varepsilon(x) := \begin{cases} 1, & |x| \ge \varepsilon,\\ \frac{1}{2\varepsilon}, & |x| < \varepsilon. \end{cases}$$

We define densities $g_\varepsilon : J \times \mathbb{R} \to \mathbb{R}$ by

$$g_\varepsilon(x,\xi) := f\left(\frac{\xi}{a_\varepsilon(x)}\right) a_\varepsilon(x) = \begin{cases} f(\xi), & |x| \ge \varepsilon,\\ \frac{1}{2\varepsilon} f(2\varepsilon\xi), & |x| < \varepsilon. \end{cases}$$

Then the families $\{f_\varepsilon\}_{\varepsilon>0}$ and $\{g_\varepsilon\}_{\varepsilon>0}$ are equivalent since for any $R>0$

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{-1}^{1}\sup_{|\xi|\le R}|f_\varepsilon(x,\xi)-g_\varepsilon(x,\xi)|\,\mathrm{d}x &= \limsup_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\sup_{|\xi|\le R}|f(\xi)-\tfrac{1}{2\varepsilon}f(2\varepsilon\xi)|\,\mathrm{d}x\\ &\le \limsup_{\varepsilon\to 0}2\varepsilon\sup_{|\xi|\le R}(|\xi|-1)\\ &=0.\end{aligned}$$

Let us denote $\lambda_\varepsilon := a_\varepsilon\mathcal{L}^1\in M(J)$. For any $u\in W^{1,1}(J)$, we may write

$$Du=u'\mathcal{L}^1=\frac{u'}{a_\varepsilon}\lambda_\varepsilon.$$

Hence, the corresponding functionals $\mathcal{G}_\varepsilon$ have on $W^{1,1}(J)$ the following representation

$$\mathcal{G}_\varepsilon(u)=\int_{-1}^{1}g_\varepsilon(x,u'(x))\,\mathrm{d}x=\int_{-1}^{1}f\left(\tfrac{\mathrm{d}Du}{\mathrm{d}\lambda_\varepsilon}(x)\right)\mathrm{d}\lambda_\varepsilon(x).$$

Since

$$\lambda_\varepsilon\overset{*}{\rightharpoonup}\lambda:=\delta_0+\mathcal{L}^1\quad\text{in } M(J),$$

it follows from [11, Theorem 2.2] that $\mathcal{G}_\varepsilon$ $\Gamma(L^1)$-converges to $\mathcal{G}_0$ where

$$\mathcal{G}_0(u)=\int_{-1}^{1}f\left(\tfrac{\mathrm{d}(D^a_\lambda u)}{\mathrm{d}\lambda}(x)\right)d\lambda(x)+\int_{-1}^{1}f^\infty\left(\tfrac{\mathrm{d}(D^s_\lambda u)}{\mathrm{d}|D^s_\lambda u|}(x)\right)\mathrm{d}|D^s_\lambda u|(x)$$

if $u\in BV(J)$ and ∞ otherwise with $D^a_\lambda u$ and $D^s_\lambda u$ being the absolutely continuous resp. singular part of Du with respect to λ. We compare these two decompositions

$$Du=u'\mathcal{L}^1+D^s u=\tfrac{\mathrm{d}(D^a_\lambda u)}{\mathrm{d}\lambda}(\mathcal{L}^1+\delta_0)+D^s_\lambda u$$

and arrive at

$$\tfrac{\mathrm{d}(D^a_\lambda u)}{\mathrm{d}\lambda}=u'\quad\mathcal{L}^1\text{-a.e.},\quad \tfrac{\mathrm{d}(D^a_\lambda u)}{\mathrm{d}\lambda}(0)=Du(\{0\})\quad\text{and}\quad (D^s u)\lfloor_{J\setminus\{0\}}=(D^s_\lambda u)\lfloor_{J\setminus\{0\}}.$$

Therefore

$$\mathcal{G}_0(u)=\int_{-1}^{1}f(u'(x))\,\mathrm{d}x+f(Du(\{0\}))+2|D^s_\lambda u|(J).$$

Hence,

$$\begin{aligned}\mathcal{F}_0(u)&=\int_{-1}^{1}f(u'(x))\,\mathrm{d}x+2|D^s u|(J),\\ \mathcal{G}_0(u)&=\int_{-1}^{1}f(u'(x))\,\mathrm{d}x+f(Du(\{0\}))+2|D^s_\lambda u|(J).\end{aligned}$$

If we choose $u:=1_{(0,1)}$, we have $Du=\delta_0$ and

$$u'=0\quad\mathcal{L}^1\text{-a.e.},\quad D^s u=\delta_0,\quad D^a_\lambda u=\delta_0,\quad D^s_\lambda u=0.$$

Hence

$$\mathcal{F}_0(1_{(0,1)})=2\neq 1=\mathcal{G}_0(1_{(0,1)}).$$

This example shows also another important issue (which is the reason of the discussion in [4]). In the case $p > 1$, the limiting functional even has a density, i.e., it is given by some Borel function $\varphi : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ with the same growth properties such that the values on $W^{1,p}(\Omega;\mathbb{R}^m)$ are given by $\int_\Omega \varphi(x, \nabla u(x))\,\mathrm{d}x$ (and ∞ elsewhere). For $p = 1$, it is still true that there is such a density that determines the values on $W^{1,1}(\Omega;\mathbb{R}^m)$ (see in [10, Theorem 12.5] or [13, Teorema]). However, it is in general wrong that the values on $BV(\Omega;\mathbb{R}^m)$ are given by

$$\int_\Omega \varphi(x, \nabla u(x))\,\mathrm{d}x + \int_\Omega \varphi^\infty\left(x, \tfrac{\mathrm{d}(D^s u)}{\mathrm{d}|D^s u|}(x)\right)\mathrm{d}|D^s u|(x).$$

Indeed, in the example above the density on $W^{1,1}(J)$ is $\boldsymbol{f}$ in both cases. However, the formula above yields $\mathcal{F}_0$. The limiting functional $\mathcal{G}_0$ for the family $\{\mathcal{G}_\varepsilon\}_{\varepsilon>0}$ has a different structure.

3 Γ-Closure

The counterexample in Sect. 2 indicates that the problem occurs when dealing with families whose difference grows linearly for large X at least on some set of x. The equivalence condition from Definition 1.2 does not exclude such behavior, and consequently, we have to impose an additional condition.

Theorem 3.1 *Let functions $f_\varepsilon^{(j)} : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $j \in \mathbb{N} \cup \{\infty\}$, $\varepsilon > 0$, uniformly fulfill the standard linear growth condition, and define*

$$\mathcal{F}_\varepsilon^{(j)}(u) := \begin{cases} \int_\Omega f_\varepsilon^{(j)}(x, \nabla u(x))\,dx, & u \in W^{1,1}(\Omega;\mathbb{R}^m), \\ \infty, & u \in L^1(\Omega;\mathbb{R}^m) \setminus W^{1,1}(\Omega;\mathbb{R}^m). \end{cases}$$

Assume that

(a) *For each $j \in \mathbb{N}$ the Γ-limit $\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(j)} =: \mathcal{F}_0^{(j)}$ exists,*
(b) *The families $\left\{\{f_\varepsilon^{(j)}\}_{\varepsilon>0}\right\}_{j\in\mathbb{N}}$ and $\{f_\varepsilon^{(\infty)}\}_{\varepsilon>0}$ are equivalent on Ω,*
(c) *For*

$$r_\varepsilon^{(j)}(R) := \operatorname*{ess\,sup}_{x\in\Omega} \sup_{|X|\geq R} \frac{|f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|}{|X|}$$

it holds

$$\lim_{R\to\infty} \limsup_{j\to\infty} \limsup_{\varepsilon\to 0} r_\varepsilon^{(j)}(R) = 0.$$

Then also $\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\infty)} =: \mathcal{F}_0^{(\infty)}$ exists. It is the pointwise and the Γ-limit of $\mathcal{F}_0^{(j)}$ as $j \to \infty$:

$$\mathcal{F}_0^{(\infty)} = \lim_{j\to\infty} \mathcal{F}_0^{(j)} = \Gamma(L^1)\text{-}\lim_{j\to\infty} \mathcal{F}_0^{(j)}.$$

Remark 3.2 (a) The equivalence condition is a sort of combination of the local uniform convergence in X and the L^1-convergence in x. We now add a uniform convergence of the slopes for large X and the L^∞-convergence in x.
(b) The assumption (c) is met if for some $\delta \in (0, 1)$ and $\gamma > 0$ it holds for all $j \in \mathbb{N}$ and $\varepsilon > 0$

$$|f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)| \leq \gamma |X|^{1-\delta}$$

for a.e. $x \in \Omega$ and $X \in \mathbb{R}^{m\times n}$. An analogous condition was already introduced when dealing with the relaxation for $p = 1$, see, e.g., [3, Section 3] and [5, Section 4].

(c) Although the recession function plays an important role in the relaxation, it does not suffice to impose just their convergence in some sense. Bear in mind that in our counterexample they even coincide, but still the statement does not hold.

For the proof of Theorem (3.1), we use the same strategy as in [16] incorporating the additional assumption on the behavior for large X.

Proof First we assume that

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\infty)} =: \mathcal{F}_0^{(\infty)}$$

exists. This will be justified in Step 3. From the growth assumption, it follows immediately that the domain of $\mathcal{F}_0^{(\infty)}$ is $BV(\Omega; \mathbb{R}^m)$ with

$$\alpha|Du|(\Omega) - \beta|\Omega| \le \mathcal{G}(u) \le \beta(|\Omega| + |Du|(\Omega))$$

for $u \in BV(\Omega; \mathbb{R}^m)$.

Step 1: For $u \in L^1(\Omega; \mathbb{R}^m)$, we claim that

$$\limsup_{j\to\infty} \mathcal{F}_0^{(j)}(u) \le \mathcal{F}_0^{(\infty)}(u).$$

By our assumption, this is obvious if $u \in L^1(\Omega; \mathbb{R}^m) \setminus BV(\Omega; \mathbb{R}^m)$.

Let us take arbitrary $u \in BV(\Omega; \mathbb{R}^m)$. Fix any $\eta > 0$, and choose a sequence $\{u_\varepsilon\}_\varepsilon \subset L^1(\Omega; \mathbb{R}^m)$ such that

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega; \mathbb{R}^m) \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\infty)}(u_\varepsilon) = \mathcal{F}_0^{(\infty)}(u).$$

Such sequence surely exists as $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(\infty)} = \mathcal{F}_0^{(\infty)}$. We may suppose that $\{\mathcal{F}_\varepsilon^{(\infty)}(u_\varepsilon)\}_\varepsilon$ is bounded and consequently $\{u_\varepsilon\}_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^m)$ with $\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1} =: B < \infty$. We choose $M > 0$ such that

$$\limsup_{j\to\infty} \limsup_{\varepsilon\to 0} r_\varepsilon^{(j)}(M) < \frac{\eta}{B},$$

and define the sets $E_\varepsilon := \{x \in \Omega : |\nabla u_\varepsilon(x)| \ge M\}$. After splitting

$$\mathcal{F}_\varepsilon^{(\infty)}(u_\varepsilon) = \int_{\Omega\setminus E_\varepsilon} f_\varepsilon^{(\infty)}(x, \nabla u_\varepsilon(x))\, \mathrm{d}x + \int_{E_\varepsilon} f_\varepsilon^{(\infty)}(x, \nabla u_\varepsilon(x))\, \mathrm{d}x,$$

we bound both terms from below by using $f_\varepsilon^{(j)}$. Obviously

$$\begin{aligned}\int_{\Omega\setminus E_\varepsilon} f_\varepsilon^{(\infty)}(x, \nabla u_\varepsilon(x))\, \mathrm{d}x \ge& \int_{\Omega\setminus E_\varepsilon} f_\varepsilon^{(j)}(x, \nabla u_\varepsilon(x))\, \mathrm{d}x \\ &- \int_\Omega \sup_{|X|\le M} |f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|\, \mathrm{d}x,\end{aligned}$$

and furthermore

$$\begin{aligned}
&\int_{E_\varepsilon} f_\varepsilon^{(\infty)}(x, \nabla u_\varepsilon(x))\,\mathrm{d}x \\
&\quad \geq \int_{E_\varepsilon} f_\varepsilon^{(j)}(x, \nabla u_\varepsilon(x))\,\mathrm{d}x - \int_{E_\varepsilon} |f_\varepsilon^{(j)}(x, \nabla u_\varepsilon(x)) - f_\varepsilon^{(\infty)}(x, \nabla u_\varepsilon(x))|\,\mathrm{d}x \\
&\quad \geq \int_{E_\varepsilon} f_\varepsilon^{(j)}(x, \nabla u_\varepsilon(x))\,\mathrm{d}x - \int_{E_\varepsilon} r_\varepsilon^{(j)}(M)|\nabla u_\varepsilon(x)|\,\mathrm{d}x \\
&\quad \geq \int_{E_\varepsilon} f_\varepsilon^{(j)}(x, \nabla u_\varepsilon(x))\,\mathrm{d}x - B\, r_\varepsilon^{(j)}(M).
\end{aligned}$$

Hence,

$$\mathcal{F}_\varepsilon^{(\infty)}(u_\varepsilon) \geq \mathcal{F}_\varepsilon^{(j)}(u_\varepsilon) - \int_\Omega \sup_{|X|\leq M} |f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|\,\mathrm{d}x - B\, r_\varepsilon^{(j)}(M).$$

Taking $\liminf$ as $\varepsilon \to 0$ on both sides of the inequality above and employing the lim inf-inequality for $\Gamma(L^1)$- $\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(j)} = \mathcal{F}_0^{(j)}$ yields

$$\mathcal{F}_0^{(\infty)}(u) \geq \mathcal{F}_0^{(j)}(u) - \limsup_{\varepsilon\to 0} \int_\Omega \sup_{|X|\leq M} |f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|\,\mathrm{d}x - \limsup_{\varepsilon\to 0}\ B\, r_\varepsilon^{(j)}(M).$$

By sending also $j \to \infty$, we arrive at

$$\mathcal{F}_0^{(\infty)}(u) \geq \limsup_{j\to\infty} \mathcal{F}_0^{(j)}(u) - \limsup_{j\to\infty}\ \limsup_{\varepsilon\to 0} \int_\Omega \sup_{|X|\leq M} |f_\varepsilon^{(j)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|\,\mathrm{d}x - \eta.$$

The claim now follows from the equivalence of the families and the arbitrariness of η.

Step 2: Lower bound. We claim that

$$\liminf_{j\to\infty} \mathcal{F}_0^{(j)}(u_j) \geq \mathcal{F}_0^{(\infty)}(u)$$

whenever $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$.

Suppose

$$u_j \to u \quad \text{in } L^1(\Omega; \mathbb{R}^m) \quad \text{and} \quad \liminf_{j\to\infty} \mathcal{F}_0^{(j)}(u_j) < \infty.$$

From

$$\lim_{R\to\infty}\ \limsup_{j\to\infty}\ \limsup_{\varepsilon\to 0}\ r_\varepsilon^{(j)}(R) = 0,$$

it follows that there is an increasing unbounded sequence $\{M_k\}_{k\in\mathbb{N}}$ such that

$$\limsup_{j\to\infty}\ \limsup_{\varepsilon\to 0}\ r_\varepsilon^{(j)}(M_k) \leq \tfrac{1}{k}.$$

We may find a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that

- all $\mathcal{F}_0^{(j_k)}(u_{j_k})$ are finite with $\lim_{k\to\infty} \mathcal{F}_0^{(j_k)}(u_{j_k}) = \liminf_{j\to\infty} \mathcal{F}_0^{(j)}(u_j)$,
- $\limsup_{\varepsilon\to 0} \int_\Omega \sup_{|X|\leq M_k} |f_\varepsilon^{(j_k)}(x, X) - f_\varepsilon^{(\infty)}(x, X)|\,\mathrm{d}x \leq \frac{1}{k}$,
- $\limsup_{\varepsilon\to 0}\ r_\varepsilon^{(j_k)}(M_k) \leq \frac{1}{k} + \limsup_{j\to\infty}\ \limsup_{\varepsilon\to 0}\ r_\varepsilon^{(j)}(M_k) \leq \frac{2}{k}$.

From

$$\infty > \sup_{k\in\mathbb{N}} \mathcal{F}_0^{(j_k)}(u_{j_k}) \geq \alpha \sup_{k\in\mathbb{N}} |Du_{j_k}|(\Omega) - \beta|\Omega|$$

it follows that $\{u_{j_k}\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega;\mathbb{R}^m)$. Then we choose ε_k (with $\varepsilon_k \searrow 0$) so small that there is a $w_k \in L^1(\Omega;\mathbb{R}^m)$ with

- $\|w_k - u_{j_k}\|_{L^1} \leq \frac{1}{j_k}$,
- $\mathcal{F}_0^{(j_k)}(u_{j_k}) + \frac{1}{j_k} \geq \mathcal{F}_{\varepsilon_k}^{(j_k)}(w_k)$,
- $\int_\Omega \sup_{|X|\leq M_k} |f_{\varepsilon_k}^{(j_k)}(x,X) - f_{\varepsilon_k}^{(\infty)}(x,X)|\,\mathrm{d}x \leq \frac{2}{k}$,
- $r_{\varepsilon_k}^{(j_k)}(M_k) \leq \frac{3}{k}$.

Notice that $\mathcal{F}_{\varepsilon_k}^{(j_k)}(w_k)$ are finite. Due to our construction, the sequence $\{w_k\}_{k\in\mathbb{N}}$ lies in $W^{1,1}(\Omega;\mathbb{R}^m)$, is there bounded and converges in $L^1(\Omega;\mathbb{R}^m)$ toward u as well. We proceed similarly as in Step 1, the only difference being that we now pass in the superscript from $\boldsymbol{j}$ to ∞. Define

$$E_k := \{x \in \Omega : |\nabla w_k| \geq M_k\}.$$

Then

$$\begin{aligned}
\mathcal{F}_{\varepsilon_k}^{(j_k)}(w_k) &= \int_{\Omega\setminus E_k} f_{\varepsilon_k}^{(j_k)}(x,\nabla w_k(x))\,\mathrm{d}x + \int_{E_k} f_{\varepsilon_k}^{(j_k)}(x,\nabla w_k(x))\,\mathrm{d}x \\
&\geq \int_{\Omega\setminus E_k} f_{\varepsilon_k}^{(\infty)}(x,\nabla w_k(x))\,\mathrm{d}x - \tfrac{2}{k} + \\
&\quad + \int_{E_k} f_{\varepsilon_k}^{(\infty)}(x,\nabla w_k(x))\,\mathrm{d}x - \tfrac{3}{k}\int_{E_k} |\nabla w_k(x)|\,\mathrm{d}x \\
&\geq \mathcal{F}_{\varepsilon_k}^{(\infty)}(w_k) - \tfrac{2}{k} - \tfrac{3}{k}\|\nabla w_k\|_{L^1}.
\end{aligned}$$

From the construction, the last inequality and the lim inf-inequality for $\{\mathcal{F}_\varepsilon^{(\infty)}\}_\varepsilon$, it follows

$$\liminf_{j\to\infty} \mathcal{F}_0^{(j)}(u_j) = \lim_{k\to\infty} \mathcal{F}_0^{(j_k)}(u_{j_k}) \geq \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}^{(j_k)}(w_k) \geq \limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}^{(\infty)}(w_k) \geq \mathcal{F}_0^{(\infty)}(u).$$

The inequalities from Steps 1 and 2 yield

$$\lim_{j\to\infty} \mathcal{F}_0^{(j)}(u) = \mathcal{F}_0^{(\infty)}(u)$$

for each $u \in L^1(\Omega;\mathbb{R}^m)$.

Step 3: Justification of our assumption.

Let us not assume a priori that $\{\mathcal{F}_\varepsilon^{(\infty)}\}_{\varepsilon>0}$ Γ-converges. We know, however, that for every subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ there exists a further subsequence $\{\varepsilon_{k_i}\}_{i\in\mathbb{N}}$ such that

$$\Gamma(L^1)\text{-}\lim_{i\to\infty} \mathcal{F}_{\varepsilon_{k_i}}^{(\infty)} =: \mathcal{F}_0^{(\infty)}$$

exists (see Appendix). From Steps 1 and 2, it follows that $\mathcal{F}_0^{(\infty)}$ does not depend on the particular subsequence $\{\varepsilon_{k_i}\}_{i\in\mathbb{N}}$. By the Urysohn property for Γ-limits, it follows that $\Gamma(L^1)-\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon^{(\infty)} = \mathcal{F}_0^{(\infty)}$. □

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Γ-Convergence

Here we recall the definition and some important properties of Γ-convergence that are used frequently in our proofs. They are taken from [9, Chapter 1].

Definition A.1 Let $\mathcal{F}_j : M \to [-\infty, \infty]$, be functionals on a metric space (M, d). We say that the sequence $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ Γ-*converges* at $x \in M$ to some $\lambda \in [-\infty, \infty]$, and denote this by

$$\lambda = \Gamma(d) - \lim_{j\to\infty} \mathcal{F}_j(x),$$

if and only if the following conditions are satisfied:

(a) (lim inf-inequality:) If $x_j \to x$ in M, then $\liminf_{j\to\infty} \mathcal{F}_j(x_j) \geq \lambda$.
(b) (Existence of a recovery sequence:) There exists a sequence $x_j \to x$ in M such that $\lim_{j\to\infty} \mathcal{F}_j(x_j) = \lambda$.

Moreover, if for a functional $\mathcal{F}_\infty : M \to [-\infty, \infty]$, it holds $\Gamma(d)\text{-}\lim_{j\to\infty} \mathcal{F}_j(x) = \mathcal{F}_\infty(x)$ for every $x \in M$, then we say that $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ Γ-*converges* to $\mathcal{F}_\infty$.

The following theorem shows the importance of the Γ-convergence in the calculus of variations.

Theorem A.2 *Let us have functionals* $\mathcal{F}_j : M \to (-\infty, \infty]$, $j \in \mathbb{N} \cup \{\infty\}$. *Suppose*

(a) *There exists a compact set* $K \subset M$ *with* $\inf_{x\in K} \mathcal{F}_j(x) = \inf_{x\in M} \mathcal{F}_j(x)$ *for all* $j \in \mathbb{N}$,
(b) $\Gamma(d)\text{-}\lim_{j\to\infty} \mathcal{F}_j = \mathcal{F}_\infty$.

Then

$$\exists \min_{x\in M} \mathcal{F}_\infty(x) = \lim_{j\to\infty} \inf_{x\in M} \mathcal{F}_j(x).$$

Moreover, if $\{x_j\}_{j\in\mathbb{N}}$ *is a precompact sequence such that* $\lim_{j\to\infty} \mathcal{F}_j(x_j) = \lim_{j\to\infty} \inf_{x\in M} \mathcal{F}_j(x)$, *then every limit of a subsequence of* $\{x_j\}_{j\in\mathbb{N}}$ *is a minimum point for* $\mathcal{F}_\infty$.

Below we state some useful properties:

- Γ-limits are always lower semicontinuous.
- For a given functional $\mathcal{F}$, its lower semicontinuous envelope is called the *relaxed functional*. It is the Γ-limit of the corresponding constant family.
- Γ-convergence possesses the Urysohn property. Namely, $\lambda = \Gamma(d)\text{-} \lim_{j\to\infty} \mathcal{F}_j(x)$ if and only if for every subsequence $\{\mathcal{F}_{j_k}\}_{k\in\mathbb{N}}$ there exists a further subsequence $\{\mathcal{F}_{j_{k_l}}\}_{l\in\mathbb{N}}$ such that $\lambda = \Gamma(d)\text{-} \lim_{l\to\infty} \mathcal{F}_{j_{k_l}}(x)$.
- On a separable metric space every sequence of functionals always contains at least a subsequence that Γ-converges.

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